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Fuzzy Quantum Logic I

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The paper gives a review of the application of fuzzy set ideas in quantum logics. After a brief introduction to the fuzzy set theory, the historical development of the main attempts to utilize fuzzy set ideas in quantum logics are presented. Results of investigations of all major researchers (except the Italian group discussed elsewhere), who work or worked in the field, are discussed.

1. INTRODUCTION

Fuzzy set theory is a very young branch of mathematics. It is only 27 years old, so it is less than half the age of quantum mechanics. Moreover, despite the fact that there are more than 20,000 people all over the world working in this field and several international journals devoted to it, fuzzy set theory is still fighting for its proper evaluation by traditionally oriented mathematicians. Therefore, it is not surprising that the possibility of applying fuzzy set theory in the foundations of quantum mechanics is far from being widely recognized. Nevertheless, I was able to collect data about more than 100 papers written on this subject by the members of two large groups of researchers: the Slovak group, consisting of 15 persons, and the Italian group consisting of 7 persons, and also several individuals: Robin Giles, the late Wawrzyniec Guz, myself, and also some other researchers who are not continuously active in the field. In the present paper, after short introduction to the fuzzy set theory, I review papers in which the notion of a fuzzy set is used explicitly, i.e., papers of the Slovak group, Robin Giles, and myself, as well as some older and recent papers of other researchers who work in the field occasionally. Review of achievements of the Italian group is left to Cattaneo (1993).

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2. FUZZY SETS

In the so-called naive set theory (which, I dare say, is sufficient for all physical purposes) the notion of a set is a primitive notion. Any specific set, e.g., the set of *members of the International Quantum Structures Association*, is defined by a predicate which unambiguously divides all objects into two classes: objects that belong to the set, and objects that do not belong to the set and form its complement. However, the opposite is not true: there are predicates which are not precise enough to define a set unambiguously. For example, let us try to define a set consisting of *people interested in quantum structures*, or a set consisting of *ripe apples*. Such a situation is encountered even in traditional mathematics: please ask someone to mark on the real line sets of numbers defined by predicates: *close to zero*, *approximately equal to ten*, or *much bigger than one*. Of course, in any such situation it is possible to draw in a more or less arbitrary way a sharp borderline by saying, for example, "the set A of numbers much bigger than one consists, by definition, of numbers bigger than, or equal to 10." But then what about 9.999999? And why should 10, not 8, 13, or 100 mark the borderline? Moreover, most probably everyone would agree that $100 \in A$ while $2 \notin A$, but what about intermediate numbers? Does the number 5 belong to A or not, and if not, does it belong to the complement of A?

Such considerations led Zadeh (1965) to introduce the notion of a *fuzzy set* $\mathscr{A}$ as being characterized by a *membership function* $\mu_{\mathscr{A}}$ whose value $\mu_{\mathscr{A}}(x) \in [0, 1]$ defines the *degree of membership of x in $\mathscr{A}$*.

Remarks

1. Throughout the paper capital script letters denote genuine fuzzy sets as well as traditional (*crisp*) sets, which are special cases of fuzzy sets. Capital italic letters are reserved for crisp sets.

2. The notion of a fuzzy set can be introduced formally in an axiomatic way by adopting the set of axioms of Zermelo–Fraenkel type (Chapin, 1974, 1975) or Gödel–Bernays type (Novak, 1980). However, the original intuitive approach of Zadeh (1965) seems to be sufficient for the purpose of the present paper, as well as for all practical purposes.

It can be easily noticed (see, e.g., Giles, 1976) that fuzzy sets are related to many-valued logics in the same way as traditional sets are related to two-valued logic. Precisely: if we apply two-valued logic to evaluate the truth-value of a sentence "x belongs to A," the possible truth-values are exclusively 0 and 1 and the set A is crisp, while if we apply many-valued logic, we obtain a fuzzy set.

Defining fuzzy sets by their membership functions proved to be very effective (some authors even identify these two notions, saying: "a fuzzy set in the universe U is an element of $[0, 1]^U$") and allowed Zadeh to define equality, inclusion, union, intersection, and complement of fuzzy sets in the following way:

$$\mathscr{A} = \mathscr{B} \quad \text{iff for all } x \in U, \quad \mu_{\mathscr{A}}(x) = \mu_{\mathscr{B}}(x) \tag{1}$$

$$\mathscr{A} \subseteq \mathscr{B} \quad \text{iff for all } x \in U, \quad \mu_{\mathscr{A}}(x) \leq \mu_{\mathscr{B}}(x) \tag{2}$$

$$\mathscr{A} \cup \mathscr{B} = \mathscr{C} \quad \text{iff for all } x \in U, \quad \mu_{\mathscr{C}}(x) = \max[\mu_{\mathscr{A}}(x), \mu_{\mathscr{B}}(x)] \tag{3}$$

$$\mathscr{A} \cap \mathscr{B} = \mathscr{C} \quad \text{iff for all } x \in U, \quad \mu_{\mathscr{C}}(x) = \min[\mu_{\mathscr{A}}(x), \mu_{\mathscr{B}}(x)] \tag{4}$$

$$\mathscr{A}' = \mathscr{B} \quad \text{iff for all } x \in U, \quad \mu_{\mathscr{B}}(x) = 1 - \mu_{\mathscr{A}}(x) \tag{5}$$

The symbols $\cup$ and $\cap$ are used throughout the paper to denote Zadeh union and intersection. However, it should be noticed that if sets $\mathscr{A}$ and $\mathscr{B}$ are crisp, $\cup$ and $\cap$ coincide with traditional set-theoretic union and intersection.

As was noticed by Giles (1976), definitions (3) and (4) can be treated as generated by connectives "or" and "and" used by Łukasiewicz (1970) already in 1920s in his studies of multiple-valued logics:

$$\tau(p \text{ or } q) = \max[\tau(p), \tau(q)] \tag{6}$$

$$\tau(p \text{ and } q) = \min[\tau(p), \tau(q)] \tag{7}$$

where $\tau(p)$ denotes truth-value of the sentence p. However, contrary to two-valued logic, there is no unique many-valued logic. If we replace formulas (6) and (7) by the following formulas, studied as well by Łukasiewicz (1970),

$$\tau(p \text{ or } q) = \min[\tau(p) + \tau(q), 1] \tag{8}$$

$$\tau(p \text{ and } q) = \max[\tau(p) + \tau(q) - 1, 0], \tag{9}$$

we obtain another multiple-valued logic, whose connectives (8) and (9) give rise to the following operations on fuzzy sets:

$$\mathscr{A} \oplus \mathscr{B} = \mathscr{C} \quad \text{iff for all } x \in U, \quad \mu_{\mathscr{C}}(x) = \min[\mu_{\mathscr{A}}(x) + \mu_{\mathscr{B}}(x), 1] \tag{10}$$

$$\mathscr{A} \odot \mathscr{B} = \mathscr{C} \quad \text{iff for all } x \in U, \quad \mu_{\mathscr{C}}(x) = \max[\mu_{\mathscr{A}}(x) + \mu_{\mathscr{B}}(x) - 1, 0] \tag{11}$$

called, respectively, *bold union* and *bold intersection* by Giles (1976). Of course, as in the case of Zadeh operations, Giles union and intersection coincide with set-theoretic operations when sets $\mathscr{A}$ and $\mathscr{B}$ are crisp.

Both standard (Zadeh) and bold (Giles) union and intersection combined with the standard fuzzy complement (5) fulfill De Morgan's laws. Of course, since there are infinitely many pairs of many-valued connectives with this feature, accordingly there are infinitely many dual pairs of fuzzy set unions and intersections. The majority of studied fuzzy set operations can be obtained pointwisely from the so-called *triangular norms* (Menger, 1942; Schweizer and Sklar, 1960), i.e., binary operations T: $[0, 1] \times [0, 1] \to [0, 1]$ which are commutative, associative, nondecreasing in each component, and such that $T(x, 1) = x$, and their dual *conorms*, defined by

$$S(x, y) = 1 - T(1 - x, 1 - y) \tag{12}$$

The most important family of triangular norms is the so-called family of *fundamental triangular norms* studied by Frank (1979) and given by

$$T_s(x, y) = \log_s \left\{1 + \frac{(s^x - 1)(s^y - 1)}{s - 1}\right\}, \qquad s \in (0, 1) \cup (1, \infty) \tag{13}$$

which yields Zadeh and Giles intersections [and also, with the aid of (12), unions] as limit cases:

$$T_0(x, y) = \lim_{s \to 0} T_s(x, y) = \min(x, y) \tag{14}$$

$$T_\infty(x, y) = \lim_{s \to \infty} T_s(x, y) = \max(x + y - 1, 0) \tag{15}$$

As we shall see later, the notion of a general triangular norm has gained popularity as the most general tool for expressing the "physical" notion of an observable.

3. FUZZY SET IDEAS IN QUANTUM LOGICS UP TO 1987

I distinguish the year 1987 because from this year on papers dealing with the application of fuzzy set ideas to quantum logics appear continuously and it is possible to distinguish several definite groups of people continuously active in the field. Up to 1987 several papers in which authors tried to utilize fuzzy sets in the foundations of quantum mechanics were published, but were scattered throughout different, sometimes not well-known journals. Moreover, their authors, after publishing one or two papers, usually did not come back to this subject.

3.1. Prehistory

To the prehistory of attempts to utilize fuzzy set ideas in quantum logics I classify papers published before 1965, i.e., before Zadeh introduced the notion of a fuzzy set in his historic paper (Zadeh, 1965). This classification, made "*post factum*," is based on my strong belief that if these papers had been published after 1965, fuzzy sets would most probably have appeared in them explicitly.

As already mentioned, Łukasiewicz's many-valued logics form a natural logical basis for the fuzzy set theory. Therefore, all work in which the utilization of many-valued logics in the quantum domain was considered, such as Reichenbach (1944), belong to the prehistory of fuzzy quantum logics.

I mention here also a short paper by Frink (1938), despite the fact that Frink separately describes algebras of propositions generated by Łukasiewicz's logic and by the Birkhoff and von Neumann (1936) logic of quantum mechanics. Nevertheless, a short comparison of the properties of different non-Boolean algebras of propositions given by Frink clearly shows that the algebra of quantum propositions is more similar to Łukasiewicz's many-valued logic endowed with connectives (8) and (9), now recognized as giving rise to Giles's connectives, than to the algebra of propositions of Heyting's logic or Łukasiewicz's logic endowed with connectives (6) and (7), which give rise to Zadeh connectives.

3.2. Prugovečki and Guz

Strictly speaking, two papers by Prugovečki (1974, 1975) only partially belong to the domain of quantum logics. However, they are the first papers known to me in which fuzzy sets were explicitly used in quantum mechanics. The idea of Prugovečki consists in the observation that "quantum phenomena display diffraction effects, which make their localization with *absolute* certainty within macroscopic regions impossible." From this statement there follows the *basic postulate* of Prugovečki (1974):

> measurement of observables in quantum mechanics yields sample points which are *normalized fuzzy sets*.

[A fuzzy set $\mathscr{A}$ is called *normalized* if $\sup \mu_{\mathscr{A}}(x) = 1$.] However, Prugovečki did not explain how these fuzzy sets could be obtained. He postulated only that (Definition 2.1. of Prugovečki, 1974):

> If $\mathscr{A}$ is the normalized fuzzy set yielded by a measurement of an observable α, then for any Borel set B in $\mathbb{R}^1$
>
> $$p_\alpha(B) = \frac{\int_B \mu_{\mathscr{A}}(x)\, dm_\alpha(x)}{(\int_{-\infty}^{+\infty} \mu_{\mathscr{A}}^2(x)\, dm_\alpha(x))^{1/2}}, \qquad [(16)]$$
>
> where
>
> $$m_\alpha(\Delta) = \int_{\Delta \cap Sc} dx + \sum_{x \in \Delta \cap Sp} \chi_{Sp}(\{x\}), \qquad [(17)]$$
>
> is the probability that a very precise measurement immediately following the measurement of α will yield a result within B. (Sc and Sp denote, respectively, continuous and point spectrum of α.)

Later on Prugovečki broadened this postulate to cover also the case of simultaneous measurement of two and more (even incompatible) observables using a kind of *probabilistic product* of fuzzy sets:

$$\mu_{\mathscr{A} \cdot \mathscr{B}}(x, y) = \mu_{\mathscr{A}}(x) \cdot \mu_{\mathscr{B}}(y) \qquad (18)$$

He stressed that despite the impossibility of arbitrary precise simultaneous measurement of α and β, the formula (18) is operationally well defined since the constituents of the right-hand side product can be independently obtained with an arbitrary degree of accuracy.

Prugovečki also studied probability measures defined on some specific families of fuzzy sets. His conclusion is summarized in the following definition (Prugovečki, 1974, Definition 3.3):

> Let $\mathfrak{F}$ be a family of fuzzy sets in $\mathbb{R}^n$ which is such that $\varnothing \in \mathfrak{F}$, and that $\mathscr{A}' \in \mathfrak{F}$ whenever $\mathscr{A} \in \mathfrak{F}$. A function $P(\mathscr{A})$ on $\mathfrak{F}$ is a *probability measure on fuzzy events* in $\mathbb{R}^n$ iff it assumes values in [0, 1], and has the following properties: (a) $P(\mathbb{R}^n) = 1$; (b) $P(\mathscr{A}') = 1 - P(\mathscr{A})$; (c) if $\mathscr{A}_1, \mathscr{A}_2, \ldots \in \mathfrak{F}$ are disjoint, then
>
> $$P\left(\bigcup_{k=1}^{\infty} \mathscr{A}_k\right) = \sum_{k=1}^{\infty} P(\mathscr{A}_k), \qquad [(19)]$$
>
> whenever $\cup_k \mathscr{A}_k$ also belongs to $\mathfrak{F}$.

Prugovečki (1974) did not explain what was meant by disjoint fuzzy sets. However, since he used only Zadeh operations, I suppose, that the disjointness of two fuzzy sets in Prugovečki (1974) should be understood as

$$\mu_{\mathscr{A} \cap \mathscr{B}}(x) = \min[\mu_{\mathscr{A}}(x), \mu_{\mathscr{B}}(x)] = 0 \qquad (20)$$

Therefore, it can be inferred from Prugovečki (1974) that his *family of fuzzy events* $\mathfrak{F}$, in order to make probabilities calculable without any restrictions, should be defined by the following conditions ($\mathbb{R}^n$ is replaced here by an abstract universe U in order to make Prugovečki's structure more similar to recently studied structures):

(i) $U \in \mathfrak{F}$.
(ii) $\mathscr{A} \in \mathfrak{F} \Rightarrow \mathscr{A}' \in \mathfrak{F}$.
(iii) $\mathscr{A}_i \cap \mathscr{A}_j = \varnothing \Rightarrow \bigcup_i \mathscr{A}_i \in \mathfrak{F}$.

It is worth mentioning that Prugovečki (1974) gave an explicit example, extracted from the Hilbert space quantum mechanics, of his probability measure on fuzzy events corresponding to measurements of incompatible observables.

In his next paper Prugovečki (1975) changed the definition of a fuzzy event in such a way that it became, in general, not a fuzzy set, but a collection of fuzzy sets. This approach, however, seems to be further away from the original spirit of fuzzy set theory, on which this paper is concentrated.

In his later papers, as well as in his book, Prugovečki (1984) did not use explicitly fuzzy sets in Zadeh's sense, since he "did not find Zadeh's ideas fruitful" (private communication, 1987), and therefore he changed the name "fuzzy phase space" to "stochastic phase space" in his later papers. However, he retained and developed the idea that numerical results of measurements in physics should be always considered together with a "confidence margin," so they are not mathematical points, but rather fuzzy sets. He introduced a notion of "stochastically extended quantum particles" which, in some sense, could be visualized as fuzzy sets of Gaussian shape. This line of thought was further developed by himself and his collaborators (T. Ali, W. Guz), and can be found in papers by Bush (1985*a,b*, 1986), Hsu and Pei (1988), Hsu and Whan (1988), and Hsu (1991).

Two papers by the late Wawrzyniec Guz (1984, 1985) are placed here, just after the works of Prugovečki, because Guz wrote them in Toronto inspired personally by Prugovečki. Nevertheless, the original field of interest of Guz was quantum logic and these two papers are the first papers known to me which can be unambiguously classified as belonging to the domain of fuzzy quantum logic.

Guz (1984) introduced the following notion of a fuzzy σ-orthoposet [notation as in Pykacz (1992)]: A *fuzzy σ-orthoposet* is a family $\mathbb{G}$ of fuzzy subsets of a universe U such that:

(i) $\mathbb{G}$ contains the empty set $\varnothing$, and the universe U.
(ii) If $\mathscr{A}, \mathscr{B} \in \mathbb{G}$ and $\mathscr{A} \subseteq \mathscr{B}$, then there exists $\mathscr{C} \ni \mathbb{G}$ such that $\mu_{\mathscr{C}} = \mu_{\mathscr{B}} - \mu_{\mathscr{A}}$.
(iii) For every sequence $\{\mathscr{A}_i\}$ in $\mathbb{G}$ such that $\sum_i \mathscr{A}_i \leq 1$, there exists $\mathscr{B} \in \mathbb{G}$ such that $\mu_{\mathscr{B}} = \sum_i \mu_{\mathscr{A}_i}$.

Guz (1984) gave several examples of fuzzy σ-orthoposets. The following one is the most interesting from the physical point of view.

Example 1. Let H be a complex Hilbert space and let $B(H)$ be the C^*-algebra of bounded operators acting on H. Then the family of fuzzy subsets of H whose membership functions are defined in the following way:

$$\mu_{\mathscr{A}}(x) = (Ax, x)/\|x\| \quad \text{if} \quad x \neq 0, \qquad \mu_{\mathscr{A}}(0) = 0 \text{ for all} \quad x \in H, \quad A \in B(H) \tag{21}$$

is a fuzzy σ-orthoposet.

In the same paper Guz (1984) also studied the possibility of the physical applications of triangular norms of Menger (1942) and Schweizer and Sklar (1960), which, as already mentioned, are now recognized as closely connected to fuzzy set operations.

Guz (1985) studied another family of fuzzy sets, which he called statistical (or fuzzy) σ-algebra. It was defined with the aid of several axioms, but in order to compare this structure with other families of fuzzy sets encountered in fuzzy quantum logics, I distinguish here three of them, using again the notation of Pykacz (1992).

A *statistical (fuzzy) σ-algebra* is a family $\mathbb{S}$ of fuzzy subsets of a universe U satisfying:

(i) $\mathbb{S}$ contains the universe U.
(ii) If $\mathscr{A} \in \mathbb{S}$, then $\mathscr{A}' \in \mathbb{S}$.
(iii) For any sequence $\{\mathscr{A}_i\}$ in $\mathbb{S}$ such that $\mathscr{A}_i \odot \mathscr{A}_j = \varnothing$ there exists $\mathscr{B} \in \mathbb{S}$ such that $\mu_{\mathscr{B}} = \sum_i \mu_{\mathscr{A}_i}$.

Let us note that the first two axioms are the same as in the definition of a family of fuzzy events of Prugovečki (1974) and nearly the same as in the definition of fuzzy σ-orthoposet of Guz (1984). As we shall see later, they also will be the same for all other families of fuzzy sets encountered in fuzzy quantum logics, which actually differ because of different versions of the third axiom. The statistical σ-algebra, however, is more specific because of the remaining axioms:

(iv) If $\mathscr{A} \odot \mathscr{B} \neq \emptyset$, then there exists $x \in U$ such that

$$\mu_{\mathscr{A}}(x) = 1 \qquad \text{and} \qquad \mu_{\mathscr{B}}(x) > 0 \tag{22}$$

(v) For each $x \in U$ there exists $\mathscr{A} \in \mathbb{S}$ such that

$$\mu_{\mathscr{A}}(x) = 1 \quad \text{and} \quad \mu_{\mathscr{A}}(y) < 1 \qquad \text{for all} \quad y \in U, \quad y \neq x \tag{23}$$

(vi) If $\mu_A(x) > 0$, then there exists one and only one element $y \in U$ such that $\mu_{\mathscr{A}}(y) = 1$, and $\mu_{\mathscr{A}}(x) = (x:y)$, where $(x:y)$, called the *transition probability from x to y*, is defined by

$$(x:y) = \inf\{\mu_{\mathscr{A}}(x) \colon \mathscr{A} \in \mathbb{S},\, \mu_{\mathscr{A}}(y) = 1\} \tag{24}$$

Guz (1985) proved that the statistical σ-algebra defined by the above axioms is an atomistic σ-orthoposet satisfying the covering law with respect to the standard fuzzy set inclusion (2) as partial order and standard fuzzy set complementation (5) as orthocomplementation. Moreover, imitating standard quantum logical procedures (e.g., Guz, 1975), he associated with his statistical σ-algebra an object called *phase geometry* obtained in a following way:

> Two elements $x, y \in U$ are called *orthogonal* and denoted $x \perp y$ if there is $\mathscr{A} \in \mathbb{S}$ such that $\mu_{\mathscr{A}}(x) = 1$ and $\mu_{\mathscr{A}}(y) = 0$. For any (crisp) subset S of U we define $S^{\perp} = \{x \in U \colon x \perp y \text{ for all } y \in S\}$ and we wrote S^{-} instead of $S^{\perp\perp}$. If $S^{-} = S$, we call the set S *$\perp$-closed*. The family of all $\perp$-closed subsets of U is called *phase geometry* associated with $\mathbb{S}$.

Guz proved that for any statistical σ-algebra its associated phase geometry endowed with the set-theoretic inclusion and complementation is an atomistic, orthocomplete, orthomodular lattice with the covering law holding in it. Later, applying "fundamental theorem of projective geometry" (e.g., Varadarajan, 1968; Maeda and Maeda, 1970), he proved that:

> If $\mathbb{S}$ is a statistical σ-algebra such that for any pair $x, y \in U$ there is a third element $z \in U$, $z \neq x, y$ such that $z \in \{x, y\}^{-} \setminus \{x, y\}$ (i.e., z is a *superposition* of x and y), and there are at least four orthogonal elements in U, then there is an inner product vector space $(V, \langle \cdot, \cdot \rangle)$ over an involutive division ring D such that $\mathbb{S}$ can be identified with a σ-orthoposet of closed subspaces of V.

3.3. Giles

I place Prof. Robin Giles among researchers working in the field of fuzzy quantum logics although it is rare that both fuzzy sets and quantum logical notions can be found in the same paper written by Giles. Therefore, in some sense, only the collection of Giles's papers (Giles, 1974, 1976, 1977, 1978, 1982) can be *jointly* classified as belonging to the domain of fuzzy quantum logics. In some papers Giles (1974, 1977, 1978) studied the structure of a formalized physical language and its relation to many-valued logic, while in others (Giles, 1976, 1982) he studied mainly relations between many-valued logic and fuzzy sets, only occasionally touching physical problems. Thus, many-valued logic is a common denominator of what I call Giles's *fuzzy quantum logic (FQL) collection of papers.*

Already in the paper "A nonclassical logic for physics" (Giles, 1974), he wrote: "*the logic* $Ł_\infty$ (i.e., infinite-valued Łukasiewicz logic) *plays the same role in the dispersive case as the classical propositional calculus does in the case of a dispersion-free language.*" Also in this paper we can find the first versions of the new notion of a *proposition* and the *dialogue interpretation of logic*, which later appeared in all the other papers of Giles's "FQL collection": "the function of a sentence is to *be asserted*, such an assertion, having the effect of expressing a *belief*. . . . and in practice such a belief may be expressed in the form of a *bet*. What is a bet? It is a particular sort of *commitment*" (Giles, 1978).

Thus, "A *proposition* is an expression whose assertion entails a definite commitment on the part of a speaker." This gives a "practical" way to define the *risk value* $\langle p \rangle$ of asserting a proposition P: "He who asserts P agrees to pay his opponent \$1 if a trial of P yields the outcome "no" . . . the risk value $\langle p \rangle$ of P for *me* denotes my expected loss if I assert P."

The meaning of compound propositions was defined in Giles (1974) in the following way:

> He who asserts $A \to B$ agrees to assert B if his opponent will assert A.
> He who asserts $\neg A$ agrees to pay \$1 to his opponent if he will assert A.
> He who asserts $A \vee B$ undertakes to assert either A or B at his own choice.
> He who asserts $A \wedge B$ undertakes to assert either A or B at his opponent's choice.

From these definitions one gets the following rules for calculating risk values of compound propositions:

$$\langle P \to Q \rangle = \sup\{0, \langle Q \rangle - \langle P \rangle\} \tag{25}$$

$$\langle \neg P \rangle = 1 - \langle P \rangle \tag{26}$$

$$\langle P \vee Q \rangle = \inf\{\langle P \rangle, \langle Q \rangle\} \tag{27}$$

$$\langle P \wedge Q \rangle = \sup\{\langle P \rangle, \langle Q \rangle\} \tag{28}$$

It is obvious that after replacing the risk value $\langle P \rangle$ by the truth value $\tau(P) = 1 - \langle P \rangle$ one obtains truth values of compound propositions of an infinite-valued logic studied by Łukasiewicz. Now, only one step is needed to pass to the fuzzy set theory, and this step was done by Giles in the paper "Łukasiewicz logic and fuzzy set theory" (Giles, 1976). This step consisted in the observation that $\mu_{\mathscr{A}}(x)$, i.e., the grade of membership of an element x to a fuzzy set $\mathscr{A}$, can be interpreted as the truth value of a proposition "$x \in \mathscr{A}$," i.e.,

$$\mu_{\mathscr{A}}(x) = \tau(\text{“}x \in \mathscr{A}\text{”}) = 1 - \langle \text{“}x \in \mathscr{A}\text{”} \rangle \tag{29}$$

Of course, expression (29) applied to propositions "$x \notin \mathscr{A}$," "$x \in \mathscr{A}$ or $x \in \mathscr{B}$," and "$x \in \mathscr{A}$ and $x \in \mathscr{B}$" gives, respectively, the Zadeh fuzzy set complement (5), union (3), and intersection (4). However, as I already mentioned, Giles (1976) noticed that there are also other possible connectives (8) and (9) in Łukasiewicz many-valued logic which give rise to "bold" operations (10) and (11) on fuzzy sets. According to my point of view, which will be presented in Section 5, these operations are particularly well suited for modeling quantum phenomena.

4. SLOVAK GROUP

The Slovak group, centered around Prof. Beloslav Riečan from Bratislava, is the largest group of researchers working in the field of fuzzy quantum logics. It consists of 13 people (among them one Vietnamese who has worked recently in Bratislava), to whom we may add two Czech researchers who have touched in their papers the same problems. The number of papers written by the Slovak group is really impressive: I was able to collect more than 80 papers. Unfortunately, only a minority of these papers have appeared in well-known journals of international range such as *Fuzzy Sets and Systems*, *Journal of Mathematical Analysis and Its Applications*, or *International Journal of Theoretical Physics*.

The first paper of the Slovak group, entitled "A new approach to some notions of statistical quantum mechanics" (Riečan, 1988), was published in

Bulletin pour les Sous-Ensembles Flous et leurs Applications in 1988, but it was noted elsewhere (Dvurečenskij and Riečan, 1991*a*) that its idea was developed already in 1986 and presented at the first IFSA-EC and EURO-WG Workshop on Progress in Fuzzy Sets in Europe held in Warsaw in autumn 1986 (Riečan and Dvurečenskij, 1988). The point of departure for this paper was the formal similarity between the notion of a state on a quantum logic and the notion of a Piasecki P-measure (Piasecki, 1985*a,b*) defined by Piasecki on his soft fuzzy σ-algebra. Let us quote here the definitions of these notions:

A *soft fuzzy σ-algebra* [called an *F-quantum space* by Riečan (1988)] is a family $\mathbb{F}$ of fuzzy subsets of a universe U containing $\varnothing$ and U, closed with respect to the standard fuzzy complementation (5) and countable Zadeh unions (3), and *not* containing the fuzzy set whose membership function equals 0.5 everywhere on U.

A *P-measure* on a soft fuzzy σ-algebra $\mathbb{F}$ is a mapping $p\colon \mathbb{F}\to[0, 1]$ such that

$$\text{(i)}\quad p(\mathscr{A}\cup\mathscr{A}')=1 \qquad \text{for all}\quad \mathscr{A}\in\mathbb{F} \tag{30}$$

$$\text{(ii)}\quad \text{if}\quad \mathscr{A}_i\subseteq\mathscr{A}_j' \quad (i\neq j), \qquad \text{then}\quad p\left(\bigcup_i \mathscr{A}_i\right)=\sum_i p(\mathscr{A}_i) \tag{31}$$

Already in this first paper of the Slovak group Riečan introduced in an obvious way the notion of an observable, "translating" it from traditional quantum logic:

An *F-observable* defined on an F-quantum space $\mathbb{F}$ (i.e., on a soft fuzzy σ-algebra) is any mapping $Z\colon \mathscr{B}(\mathbb{R}^1)\to\mathbb{F}$ such that

$$\text{(i)}\quad Z(A')=(Z(A))' \qquad \text{for every}\quad A\in\mathscr{B}(\mathbb{R}^1) \tag{32}$$

$$\text{(ii)}\quad \text{if}\quad A_n\in\mathscr{B}(\mathbb{R}^1),\quad n=1,2,\ldots,\quad \text{and}\quad A_n\cap A_m=\varnothing,$$

$$\text{then}\quad Z\left(\bigcup_n A_n\right)=\bigcup_n Z(A_n) \tag{33}$$

[The left-hand side complement in (32) is an ordinary set-theoretic complement, while the right-hand side complement of (32) is the standard fuzzy complement. The same pertains to unions in (33).]

In subsequent papers (Riečan and Dvurečenskij, 1988; Dvurečenskij and Riečan, 1988, 1991*b*; Dvurečenskij and Tirpaková, 1988) and other papers of the Slovak group too numerous to be listed here [cf. the bibliographical essay of Cattaneo *et al.*, (n.d.)] this line of investigations was continued. Notions of joint observables, sum of observables, compatibility, commensurability, and other notions typical of quantum logic issues were introduced and studied in the realm of F-quantum spaces.

Dvurečenskij and Chovanec (1988) originated another stream of papers based on the notion of a *fuzzy quantum space*, also called *type I fuzzy quantum poset* (Dvurečenskij and Riečan, 1991*a*; Lê Bá Long, n.d.) or *fuzzy quantum poset* (Dvurečenskij and Lê Bá Long, 1991). Fuzzy quantum space differs from F-quantum space since it is assumed to be closed not with respect to Zadeh unions of arbitrary families of fuzzy sets, but only with respect to Zadeh unions of families of parwise *orthogonal* fuzzy sets [also called *weakly disjoint* sets by Giles (1976) or *W-separated* sets by Piasecki (1985*a*)], i.e., such fuzzy sets that

$$\mu_{\mathscr{A}}(x) \leq 1 - \mu_{\mathscr{B}}(x) = \mu_{\mathscr{B}'}(x) \qquad \text{for all} \quad x \in U \tag{34}$$

which, using Giles's intersection, can be expressed equivalently

$$\mathscr{A} \odot \mathscr{B} = \varnothing \tag{34'}$$

It was noticed by Durečenskij and Chovanec (1988) that a fuzzy quantum space can be thought of as a "fuzzyfication" of the notion of a quantum probability space introduced by Suppes (1966) as a family of (crisp) sets closed with respect to complementation and countable unions of disjoint sets.

Again in subsequent papers which are too numerous to be listed [but can be found, as previously, in Cattaneo *et al.* (n.d.)], practically all the basic quantum logical notions were redefined in such a way that they were based on the notion of a fuzzy quantum space instead of traditional quantum logic. Also a probability theory for states and observables (e.g., laws of large numbers, martingale theorem, ergodic theorem, Radon–Nikodym theorem) has been constructed. The structure of fuzzy quantum spaces and their fuzzy observables was clarified by Dvurečenskij (n.d.), using the Loomis–Sikorski representation theorem, and by Kolesárová and Mesiar (1990; Kolesárová, 1990), using the constructive method. Their representation theorems make it possible to obtain in a simple way the majority of previously obtained results.

The notion of a fuzzy quantum space (poset) was further varied (Dvurečenskij and Riečan, 1991*a*; Lê Bá Long, n.d.) by assuming that the studied structure is closed with respect to Zadeh sums of families of *fuzzy orthogonal sets*:

$$\mathscr{A} \perp_{\mathrm{F}} \mathscr{B} \quad \text{iff} \quad \mu_{\mathscr{A} \cap \mathscr{B}}(x) \leq 0.5 \tag{35}$$

or *strongly orthogonal sets*:

$$\mathscr{A} \perp_{\mathrm{S}} \mathscr{B} \quad \text{iff} \quad \mathscr{A} \cap \mathscr{B} = \varnothing \tag{36}$$

i.e., simply: disjoint fuzzy sets.

The obtained structures were then called, respectively, *type II* and *type III fuzzy quantum posets*, and the latter structure shows some similarity to the already mentioned family of fuzzy events of Prugovečki (1974). Of course, again traditional quantum logical notions were "translated" and studied within this framework.

The typical feature of the mentioned papers of the Slovak group is that in all considered structures the standard (Zadeh) fuzzy complement (5), union (3), and, therefore, also intersection (4) were used. However, the standard fuzzy complementation is *not* an orthocomplementation in any family of fuzzy sets partially ordered by the fuzzy set inclusion (2) and endowed with Zadeh union and intersection, respectively, as supremum and infimum. To see this, it is enough to notice that for any genuine (i.e., noncrisp) fuzzy set $\mathscr{A}$

$$\mathscr{A} \cup \mathscr{A}' \neq U \tag{37}$$

$$\mathscr{A} \cap \mathscr{A}' \neq \varnothing \tag{38}$$

i.e., neither the law of excluded middle nor the law of contradiction holds in such structures. Maybe because of this fact the papers of the Slovak group, however very well elaborated mathematically, are very poor in physical examples. According to my point of view, the structures become more "physically plausible" when Giles operations are used instead of Zadeh ones. This is done in the approach which will be presented in the next section. However, several of the most recent Slovak papers are based on a quite general notion of a triangular norm (Mesiar, 1991, n.d.; Kolesárová and Riečan, n.d.), or on a notion of a general binary operation on fuzzy sets (Riečan, 1992). Such a general approach is very interesting since it covers structures obtained with the aid of Zadeh as well as of Giles operations, and, of course, also other possible specific structures obtained with the aid of other fuzzy unions and intersections. Therefore, any result obtained within such a general approach is valid in all possible specific cases, and such a general approach is a unification of many results obtained by the Slovak group and other researchers.

5. PYKACZ

My own approach to fuzzy quantum logics began in 1985 when I read the first book on fuzzy set theory and I was struck by the transparency and beauty of this idea. My previous acquaintance (Pykacz, 1983) with Mączyński's Theorem (Mączyński, 1973, 1974) allowed me to notice that with the aid of this theorem many quantum logical notions can be translated in a straightforward way into the language of fuzzy set theory. Indeed, Mączyński's Theorem says that (Mączyński, 1973):

If L is a quantum logic (i.e., σ-orthocomplete orthomodular poset) with a full set of states S, then each element $a \in L$ induces a function $\underline{a}: S \to [0, 1]$, $\underline{a}(p) = p(a)$, for all $p \in S$. The set of all such functions $\underline{L} = \{\underline{a}: a \in L\}$ satisfies the following

Orthogonality Postulate: if $\underline{a}_i + \underline{a}_j \leq 1$ for $i \neq j$, then there exists $\underline{b} \in \underline{L}$ such that $\underline{b} + \underline{a}_1 + \underline{a}_2 + \cdots = 1$.

$\underline{L}$ equipped with natural partial order, $\underline{a} \leq \underline{b} \Leftrightarrow \underline{a}(p) \leq \underline{b}(p)$ for all $p \in S$, and complementation $\underline{a}' = 1 - \underline{a}$, is isomorphic to L.

Conversely, if $\underline{L} \subseteq [0, 1]^X$ is a set of functions for which the Orthogonality Postulate holds, then it is a quantum logic with respect to natural partial order and complementation.

I noticed (Pykacz, 1987*a*) that functions $\underline{a}: S \to [0, 1]$ generated by elements of a logic can be immediately used as membership functions defining fuzzy subsets of the set of all states S of a physical system. Therefore, logics of physical systems can be thought of as families of fuzzy subsets of S, whose membership functions satisfy the Orthogonality Postulate. In subsequent papers (Pykacz, 1988, 1989, 1990) the Orthogonality Postulate was expressed in terms of the Giles union (10), intersection (11), and the standard fuzzy complementation (5), and the definition of a fuzzy quantum logic (equivalent, as it was already stated, to traditional quantum logic with a full set of probability measures) took the following form (Pykacz, 1992):

A *fuzzy quantum logic* is any family of fuzzy sets $\mathfrak{F}$ satisfying the following conditions:

(i) $\varnothing \in \mathfrak{F}$.

(ii) $\mathscr{A} \in \mathfrak{F} \Rightarrow \mathscr{A}' \in \mathfrak{F}$.

(iii) If sets $\mathscr{A}_1, \mathscr{A}_2, \ldots$ are pairwise weakly disjoint [(34), (34′)], then $\sum_i \mu_{\mathscr{A}_i} \leq 1$, and $\bigoplus_i \mathscr{A}_i \in \mathfrak{F}$.

We see that only the part of condition (iii) which says that the algebraic sum of membership functions of any sequence of pairwise weakly disjoint sets does not exceed 1 is not expressed in terms of Giles operations. In fact, this condition is neither natural nor easy to be fulfilled, but careful examination of the proof of the Mączyński Theorem shows that it is responsible for the σ-orthocompleteness of the obtained structure, and for the fact, that $\mathscr{A}' = 1 - \mathscr{A}$ is an orthocomplementation in $\mathfrak{F}$. Fortunately, my most recent results (Pykacz, n.d.-*a*), based on Mesiar's (n.d.) studies of general structures in which specific, Zadeh or Giles, connectives are replaced by general triangular norms and conorms [(12), (13)], show that the unnatural condition $\sum_i \mu_{\mathscr{A}_i} \leq 1$ can be avoided if we add the following requirement:

(iv) For any $\mathscr{A} \in \mathfrak{F}$, if $\mathscr{A} \odot \mathscr{A} = \varnothing$, then $\mathscr{A} = \varnothing$.

Or, equivalently:

(iv′) For any $\mathscr{A} \in \mathfrak{F}$, if $\mathscr{A} \subseteq \mathscr{A}'$, then $\mathscr{A} = \varnothing$.

This requirement is much more natural, since if elements of a logic are interpreted as propositions about a physical system, and partial order and orthocomplementation play, respectively, roles of implication and negation, then $\mathscr{A} \subseteq \mathscr{A}'$ would mean that a proposition implies its own negation.

Concluding, we can concisely and uniformly define a fuzzy quantum logic as a family of fuzzy sets satisfying the conditions (i), (ii), (iii′), and (iv) or (iv′), where the condition (iii′) arises from (iii) by dropping the requirement $\sum_i \mu_{\mathscr{A}_i} \leq 1$. Moreover, Mesiar's (n.d.) results indicate that fuzzy quantum logics are the only (up to an isomorphism) families of fuzzy sets equipped with pointwise-generated fuzzy connectives, which are quantum logics in the traditional sense.

Translation of the notion of a quantum logic into the language of fuzzy set theory implies changes in the interpretation of some basic entities which appear in the quantum logic approach to the foundations of quantum mechanics. For example, the number $p(a) \in [0, 1]$, $p \in S$, $a \in L$, instead of being interpreted as the probability of obtaining a positive result in an experiment testing a property a when a physical system is in a state p, can be interpreted in the following way (Pykacz, 1987*b*):

> $p(a) = \mu_{\mathscr{A}}(p)$ is the grade of membership of the state p to the (in general, fuzzy) subset $\mathscr{A}$ of S which collects all states for which the result of an experiment testing the property a is positive.

This means that the set $\mathscr{A}$ is defined by a (fuzzy) predicate: "a physical system has the property a." Equivalently, and according to the very spirit of the fuzzy set theory, we can say that

> $p(a) = \mu_{\mathscr{A}}(p)$ is the grade to which a physical system in the state p has the property a.

Continuing this line of thought, we can say that a quantum object, even before any experiment or measurement is done, actually "has" all possible properties (or all possible values of its observables), each of them to the degree allowed by suitable quantum mechanical calculations. Of course, before the experiment is completed, this statement belongs to the domain of multiple-valued logic.

Translation of quantum logical notions into the language of fuzzy set theory exhibits both striking similarities and remarkable difference between logics of quantum and classical systems. In the traditional approach, the logic of a classical system is a Boolean algebra of subsets of a phase space endowed with set-theoretic operations, while the logic of a quantum system is a lattice of subspaces (not subsets!) of a Hilbert space, and it is endowed with operations which differ from the set-theoretic ones. In the fuzzy set approach, in both cases the logic of a physical system is a family of fuzzy subsets of the set of all states S, endowed with Giles operations and the standard fuzzy complementation (Pykacz, 1987*a*,*b*, 1988, 1989, 1990, 1992). The difference between classical and quantum systems becomes clear when we restrict considerations to subsets of the set of all *pure* states P: logics of classical systems are then Boolean algebras of crisp subsets of P, while logics of quantum systems unavoidably contain genuine fuzzy (i.e., non-crisp) subsets of P, and are σ-orthocomplemented orthomodular posets (Pykacz, 1987*a*,*b*, 1988, 1989, 1990, 1992). Moreover, only within this approach is is possible to make a "smooth" transition from the quantum to the classical case by making elements of a logic less and less fuzzy.

The field of events in classical, i.e., Kolmogorovian, probability theory is a Boolean algebra. It is replaced by a lattice of closed subspaces of a Hilbert space or by a general orthomodular poset of propositions in attempts to build a non-Kolmogorovian probability theory suitable for quantum mechanics. The above-mentioned results show that the notion of a fuzzy quantum logic can be a starting point for building a unified probability theory which could cover both classical and quantum cases.

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